Transfer Function Synthesis by State Vector Feedback

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Introduction

THE theory of closed-loop pole allocation in time-invariant linear systems by state vector feedback is now well understood,[1] but the problem of allocating transfer function zeros (or equivalently, the coefficients of the numerator polynomials) as well as the poles, is currently under investigation. Chen[2] has described a sequential zero-pole placement technique, and Fallside and Patel[3] have approached the problem by using unity-rank feedback, which has been shown to be very restrictive. Wang and Desoer[4] have described a procedure for "exact model matching," but this gives little guidance as to how to proceed if the particular transfer functions desired cannot be achieved. Another approach[5], suitable for a certain class of system, uses the results of modal control theory directly.

The problem considered here is the determination of the state vector feedbacks needed to give a desired scalar transfer function between one input and one output of a system which has two inputs. The number of inputs available often is limited in practical cases, and so the system considered may be regarded as representing the most unfavorable multi-input case. The procedure gives guidance at each stage on any constraints on the design. Means are also provided for monitoring all other input/output and input/state transfer functions during the design process.

System Description

A linear system is described by the equations:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{1}$$

$$\mathbf{y} = C\mathbf{x} \tag{2}$$

where $\mathbf{x}$ is an $n \times 1$ state vector,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ is a } 2 \times 1 \text{ input vector,}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \text{ is an } m \times 1 \text{ output vector,}$$

$$B = [\mathbf{b}_1 \quad \mathbf{b}_2] \tag{3}$$

and

$$C = \begin{bmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_m^T \end{bmatrix} \tag{4}$$

The system is observable, and is completely controllable through $\mathbf{b}_2$ alone. The n-vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent.

Note: If (A, B) is controllable, $(A + BK, \mathbf{b}_2)$ can be made to be controllable by the use of suitable feedback K.[6]

Problem Statement

The problem is to find the feedback vectors $\mathbf{k}_1^T$ and $\mathbf{k}_2^T$ such that the system:

$$\dot{\mathbf{x}} = \left\{ A + [\mathbf{b}_1 \vdots \mathbf{b}_2] \begin{bmatrix} \mathbf{k}_1^T \\ \cdots \\ \mathbf{k}_2^T \end{bmatrix} \right\} \mathbf{x} + B\mathbf{u} \qquad \mathbf{y} = C\mathbf{x} \tag{5}$$

has a transfer function relating $y_1(s)$ to $u_1(s)$, given by:

$$\frac{p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_1 s + p_0}{s^n + a'_{n-1}s^{n-1} + \cdots + a_1's + a_0'} \tag{6}$$

in which the p_j, $j = 0, \ldots (n-2)$, and the a_i', $i = 0, \ldots (n-1)$, are to be given preassigned values. The value of p_{n-1} is $\mathbf{c}_1^T\mathbf{b}_1$, and cannot be assigned.

Preliminary Results

A system of the type represented in Eqs. (1) and (2) has a transfer function relating an input corresponding to a general column $\mathbf{b}$ of B to an output corresponding to a general row $\mathbf{c}^T$ of C given by:

$$\mathbf{c}^T(sI - A)^{-1}\mathbf{b} \tag{7}$$

where

$$(sI - A)^{-1} = \operatorname{adj}(sI - A)/\det(sI - A) \tag{8}$$

Now

$$\operatorname{adj}(sI - A) = Is^{n-1} + G_{n-2}s^{n-2} + \cdots + G_1 s + G_0 \tag{9}$$

where the G_j are $n \times n$ constant matrices. The G_j can always be computed in a routine manner, e.g., by the Leverrier algorithm, but in the case in which the A matrix is in the companion form A_c, the G_j have a particularly simple form, so that they can be written down. The formulas for writing these matrices are given in Appendix A. The matrix A_c for a fifth-order system is of the form:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix} \tag{10}$$

and the G_j matrices are given in full for this case in Eq. (11).

$$\left.\begin{aligned} G_4 &= I \\ G_3 &= \begin{bmatrix} a_4 & 1 & 0 & 0 & 0 \\ 0 & a_4 & 1 & 0 & 0 \\ 0 & 0 & a_4 & 1 & 0 \\ 0 & 0 & 0 & a_4 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & 0 \end{bmatrix} \\ G_2 &= \begin{bmatrix} a_3 & a_4 & 1 & 0 & 0 \\ 0 & a_3 & a_4 & 1 & 0 \\ 0 & 0 & a_3 & a_4 & 1 \\ -a_0 & -a_1 & -a_2 & 0 & 0 \\ 0 & -a_0 & -a_1 & -a_2 & 0 \end{bmatrix} \\ G_1 &= \begin{bmatrix} a_2 & a_3 & a_4 & 1 & 0 \\ 0 & a_2 & a_3 & a_4 & 1 \\ -a_0 & -a_1 & 0 & 0 & 0 \\ 0 & -a_0 & -a_1 & 0 & 0 \\ 0 & 0 & -a_0 & -a_1 & 0 \end{bmatrix} \\ G_0 &= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & 1 \\ -a_0 & 0 & 0 & 0 & 0 \\ 0 & -a_0 & 0 & 0 & 0 \\ 0 & 0 & -a_0 & 0 & 0 \\ 0 & 0 & 0 & -a_0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{11}$$

From Eqs. (7) and (9), the coefficient p_q of s^q in the transfer function numerator polynomial is given by

$$p_q = \mathbf{c}^T G_q \mathbf{b} \tag{12}$$

This relationship enables all the coefficients to be found for all the transfer functions, by the appropriate choice of $\mathbf{c}^T$, $\mathbf{b}$, and G_q. Where the A matrix is in the companion form, it is clear that the G_j are functions of the a_i, the coefficients of the characteristic equation of the A matrix. It is then possible to formulate a set of equations from which the a_i can be found so as to give desired

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coefficients p_q in a given transfer function numerator polynomial. These equations are set out in full in Eq. (13) for a system of fifth order, where $\mathbf{c}^T = [c_1 c_2 \ldots c_5]$ and $\mathbf{b} = [b_1 b_2 \ldots b_5]^T$. In Appendix B, formulas are given which enable the equations to be written down for a system of any order.

$$\left.\begin{bmatrix} (c_1b_1+c_2b_2+c_3b_3+c_4b_4) & -c_5b_4 & -c_5b_3 & -c_5b_2 & -c_5b_1 \\ (c_1b_2+c_2b_3+c_3b_4) & (c_1b_1+c_2b_2+c_3b_3) & -(c_4b_3+c_5b_4) & -(c_4b_2+c_5b_3) & -(c_4b_1+c_5b_2) \\ (c_1b_3+c_2b_4) & (c_1b_2+c_2b_3) & (c_1b_1+c_2b_2) & -(c_3b_2+c_4b_3+c_5b_4) & -(c_3b_1+c_4b_2+c_5b_3) \\ c_1b_4 & c_1b_3 & c_1b_2 & c_1b_1 & -(c_2b_1+c_3b_2+c_4b_3+c_5b_4) \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} r_3 \\ r_2 \\ r_1 \\ r_0 \end{bmatrix} \right\} \quad (13)$$

where: $r_3 = p_3-(c_1b_2+c_2b_3+c_3b_4+c_4b_5)$, $r_2 = p_2-(c_1b_3+c_2b_4+c_3b_5)$, $r_1 = p_1-(c_1b_4+c_2b_5)$, and $r_0 = p_0-c_1b_5$.

form Eqs. (7) and (9), the coefficients of the numerator polynomials of any transfer functions can be examined. Where these transfer functions relate to the input u_1, the numerators will not change when the feedback $\mathbf{k}_1{}^T$ is applied. The numerators of transfer functions relating to other inputs in general will change when $\mathbf{k}_1{}^T$ is applied. Hence, Eqs. (7) and (9) must be used accordingly.

Where the state vector is not available directly, provided that the system is observable, a degenerate observer may be used[8-10] to provide the two linear functional state-vector feedbacks required. This observer may be of very low order.

Procedure

The design proceeds in two stages. In the first stage, the feedback vector $\mathbf{k}_2{}^T$ is found in order to give the desired numerator polynomial to the transfer function:

$$\mathbf{c}_1{}^T(sI - A - \mathbf{b}_2\mathbf{k}_2{}^T)^{-1}\mathbf{b}_1 \quad (14)$$

This feedback changes the system eigenvalues to some new set. In the second stage, the feedback vector $\mathbf{k}_1{}^T$ is found in order to change the eigenvalues to the desired set corresponding to the required system poles.

Since system (1) and (2) is controllable through $\mathbf{b}_2$, it can be transformed into the companion form by the state vector transformation $\mathbf{z} = T\mathbf{x}$, so that the $\mathbf{b}_2$ vector in the $\mathbf{z}$ space becomes $T\mathbf{b}_2 = \mathbf{e}_n$, where $\mathbf{e}_n$ is a unit column n vector with a 1 in the last row, and zeros elsewhere. In this form, the feedback vector $\mathbf{k}_2{}^T T^{-1}$ can be written down to change the a_i in the last row of the A_c matrix to any desired values.

Let $\mathbf{c}^T = \mathbf{c}_1{}^T T^{-1}$ and $\mathbf{b} = T\mathbf{b}_1$ in Eqs. (13). If these equations are consistent, they may be solved for the a_i to give the desired p_q. Otherwise, row reduction of these equations will provide a set of linear constraints on the p_q which must be satisfied to give a solution.

We now proceed to find $\mathbf{k}_1{}^T$ to give the desired closed-loop poles. As is well known, the application of this feedback will not disturb the numerator polynomials of transfer functions relating to this input, which were obtained in the first stage. It is first necessary to check the pair $[(A+\mathbf{b}_2\mathbf{k}_2{}^T), \mathbf{b}_1]$ for controllability. If this test is satisfied, we may find $\mathbf{k}_1{}^T$ to give any preassigned set of closed-loop poles. Otherwise, small changes in $\mathbf{k}_2{}^T$, and hence in the p_q, must be introduced, so as to achieve controllability. The validity of this procedure is established by the following theorem.

Theorem

If the pair $(A, \mathbf{b}_2)$ is controllable, and $\mathbf{b}_1 \neq \mathbf{0}$, a vector $\mathbf{k}^T$ can be chosen with elements arbitrarily close to those of a given vector $\mathbf{k}_2{}^T$, such that $[(A+\mathbf{b}_2\mathbf{k}^T), \mathbf{b}_1]$ is controllable.

Proof: Applying to this case a lemma of Heymann,[7] there exists a vector $\mathbf{k}^T$ such that $[(A+\mathbf{b}_2\mathbf{k}^T), \mathbf{b}_1]$ is controllable. Let $\mathbf{k}^T$ be so chosen. Now change the first element of $\mathbf{k}^T$, noting that there is a finite number of values of the change which give uncontrollability. We may, therefore, choose a value which makes this element either equal to or arbitrarily close to the first element of $\mathbf{k}_2{}^T$, while preserving controllability. Repetition of this process for each element of $\mathbf{k}^T$ in turn, retaining the changed value at each step, completes the proof.

The determination of $\mathbf{k}_1{}^T$ may be achieved by again transforming to the companion form, this time the matrix $(A+\mathbf{b}_2\mathbf{k}_2{}^T)$, so that the vector $\mathbf{b}_1$ becomes $\mathbf{e}_n$. Thus, the denominator coefficients a_i' are assigned as desired.

Conclusion

The procedure gives a general method of approach for determining the state vector feedbacks required for the synthesis of the scalar transfer function. It is clearly closely related to that of Ref. 4, but gives more information at each stage of the design process. Constraints on the design are revealed, and can be allowed for at the appropriate stage. By use of relations of the form Eqs. (7) and (9), the coefficients of the numerator polynomials of any transfer functions can be examined.

Appendix A: Rules for Writing the G_j Matrices

$G_{n-1} = I_n$. Note that $a_n = 1$. For $j < n-1$; The first $(j+1)$ elements on the main diagonal are a_{j+1}, and the rest are zeros. The first $(j+1)$ elements on the ith diagonal above the main diagonal are a_{j+i+1}, and the rest are zeros. For $(j+i+1) > n$, all diagonal entries are zero. The last $(n-j-1)$ elements on the ith diagonal below the main diagonal are $-a_{j-i+1}$, and the rest are zeros. For $(j-i+1) < 0$, all the diagonal entries are zero.

Appendix B: Rules for Writing the Linear Equation Set

The general term in the ith row and the jth column in the coefficient matrix on the left-hand side of the equations generalized from Eq. (13) is:

$$\sum_{p=1}^{n-i} c_p b_{p+i-j} \qquad \text{for } j \leq i$$

and

$$\sum_{p=n-i+1}^{n} c_p b_{p+i-j} \qquad \text{for } j > i$$

The general term in the ith row on the right-hand side is:

$$q_{n-i-1} - \sum_{p=1}^{n-i} c_p b_{p+i}$$

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Elastodynamics of Cracked Structures Using Finite Elements

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WITHIN the past few years, a number of special crack-tip finite elements have been developed[1–6] for application to cracked structural components whose irregular geometry or otherwise inconvenient boundary conditions dictate a numerical analysis. Because these elements include in their formulation the $r^{-1/2}$ crack-tip stress singularity inherent to two-dimensional fracture mechanics, their incorporation in a finite-element model permits an accurate and economical assessment of the stresses near a crack tip. Specifically and most importantly, the singularity element gives directly the value of the plane-deformation stress-intensity factors K_I and K_{II} for the particular loads and crack length under consideration. All applications of this type with which the authors are familiar have been either for equilibrium problems or problems for which the inertia forces can reasonably be neglected. In this Note we present applications in elastodynamics of two singularity elements whose stiffnesses were characterized earlier by Aberson, Anderson, and Hardy.[7–8] The elements, one for problems which are symmetric with respect to the crack (so that only the crack-opening mode of deformation is operative) and one for general plane applications (involving both the opening and sliding modes), are shown in Fig. 1.

For symmetric problems only half of the region near the tip of a crack need be represented. The singularity element used for such applications is shown in Fig. 1a. It is rectangular with a 3:1 aspect ratio and has 8 nodes. The 16 displacement degrees of freedom correspond to the first 13 symmetric modes of deformation as given by Williams[9] plus the 3 rigid-body degrees of freedom. The crack tip is located midway along the bottom edge of the element at the origin of the polar coordinate system. The upper crack face extends from this point horizontally to the left. As dictated by symmetry, nodes along the prolongation of the crack are restrained against vertical displacement. For asymmetric problems the entire region in the immediate vicinity of the crack

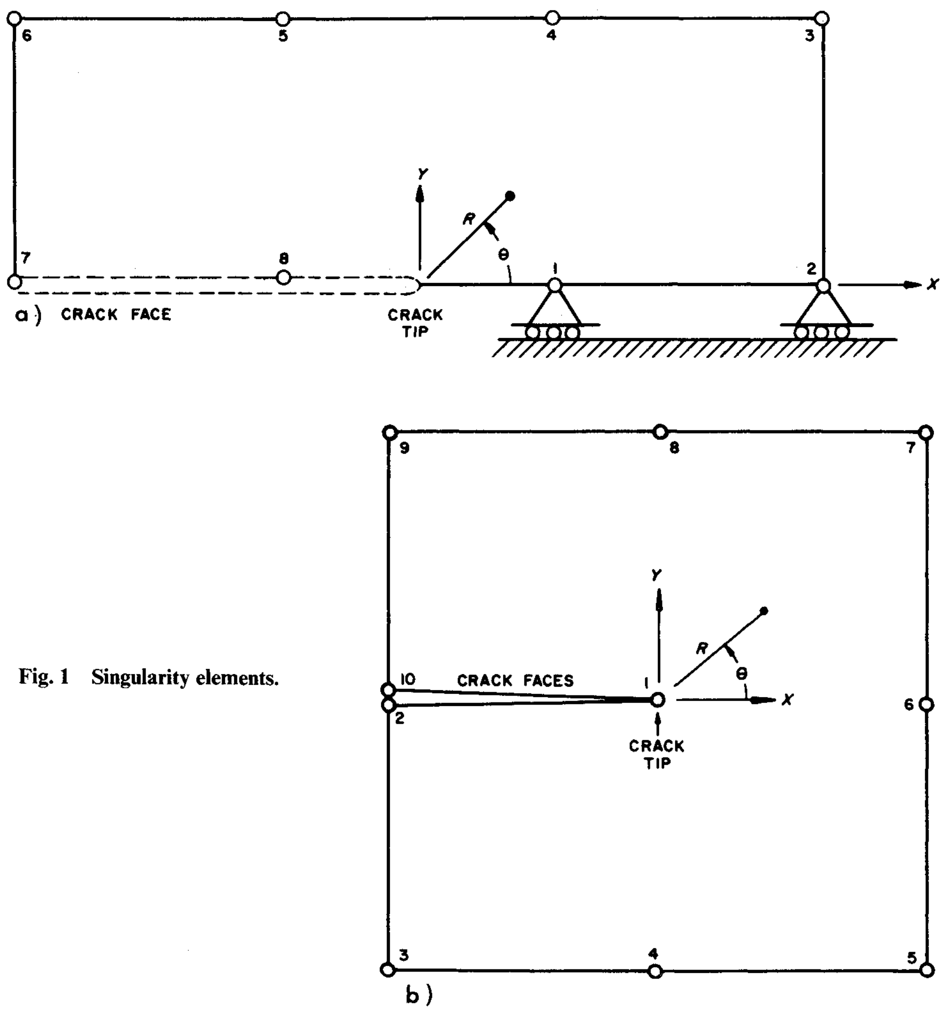

Fig. 1 Singularity elements.

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